Feb 8, 2023

Week 5

2020 B Adv. Cal II

Triple integral and double integral share the same theory.

Let B be a rectangular box $\subset \mathbb{R}^3$

$$[a,b]\times[c,d]\times[e,f].$$

Let P be a partition on B:

$$a = x_0 < x_1 < \cdots < x_n = b,$$
$$c = y_0 < y_1 < \cdots < y_m = d,$$
$$e = z_0 < z_1 < \cdots < z_l = f.$$

Sub-rectangular box

$$B_{ijk} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]\times[z_{k-1}, z_k].$$

For a function $f = f(x,y,z)$ in B, the Riemann sum of f with respect to P:

$$R(f,P) = \sum_{i,j,k} f(P_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k, \quad P_{ijk}\in B_{ijk} \text{ (tag pt)}$$

f is <u>integrable</u> if there is a number I s.t. whenever $\|P\|\to 0$, $R(f,P)$ tends to I regardless of the tag points. The number I is the <u>integral</u> of f over B, denoted by

$$\iiint_B f\,dV, \quad \iiint_B f(x,y,z)\,dV(x,y,z), \quad \text{etc.} \qquad \text{Here}$$

$$\|P\| = \max\{\Delta x_i, \Delta y_j, \Delta z_k\}$$

- continuous fcns are integrable
- more generally, piecewise continuous fcns are integrable.

Fubini's theorem For a continuous fcn f in B,

$$\iiint_B f\,dV = \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y)$$

$$= \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx, \quad \left(\begin{array}{c}\text{after applying}\\ \text{Fubini thm for}\\ \text{double integral}\end{array}\right)$$

where $R = [a,b]\times[c,d]$.

"Ideas of Pf" Take $P_{ijk} = (x_i^*, y_j^*, z_k^*)$,

$$\iiint_B f\,dV \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

$$= \sum_{i,j} \left(\sum_k f(x_i^*, y_j^*, z_k^*\,\Delta z_k\right)\Delta x_i \Delta y_j$$

$$\approx \sum_{i,j} \int_e^f f(x_i^*, y_j^*, z)\,dz\,\Delta x_i \Delta y_j$$

$$\approx \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y). \quad \square$$

Now, when f defined in a region Ω in space. Define

$$\iiint_\Omega f\,dV = \iiint_B \tilde{f}\,dV,$$ where B is a rectangular box containing Ω and the universal ext.

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z)\in\Omega \\ 0, & (x,y,z)\notin\Omega. \end{cases}$$

when Ω is of the form

$$g_1(x,y) \le z \le g_2(x,y)$$
$$(x,y) \in D,$$

we have the basic formula:

$$\boxed{\iiint_\Omega f\, dV = \iint_D \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\, dA(x,y)}.$$

the volume of Ω is defined to be

$$\iiint_\Omega 1\, dV.$$

When Ω is of the above form,

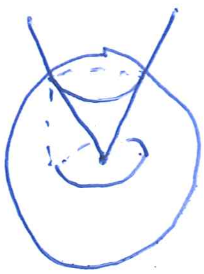
$$\iiint_\Omega 1\, dV = \iint_D \int_{g_1(x,y)}^{g_2(x,y)} 1\, dz\, dA(x,y)$$

$$= \iint_D \big(g_2(x,y) - g_1(x,y)\big)\, dA(x,y),$$

which is our old formula.

e.g. Evaluate $\iiint_\Omega z\, dV$ where Ω is the ice-cream cone bounded above by $x^2+y^2+z^2=8$ and below by $z=\sqrt{x^2+y^2}$.

the two surfaces intersect at



$$\sqrt{8-x^2-y^2} = z = \sqrt{x^2+y^2}, \text{ ie}$$

$$8-x^2-y^2 = x^2+y^2,$$

$$x^2+y^2 = 4$$

So, $D: x^2+y^2 \le 4$ and

$g_1(x,y) = \sqrt{x^2+y^2}$, $g_2(x,y) = \sqrt{8-x^2-y^2}$.

$$\iiint_\Omega z\,dV = \iint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z\,dz\,dA(x,y)$$

$$= \frac{1}{2}\iint_D (8-2(x^2+y^2))\,dA(x,y)$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (8-2r^2)\,r\,dr\,d\theta$$

$$= 2\pi\int_0^2 (4-r^2)\,r\,dr = 2\pi\left(2r^2-\frac{r^3}{3}\right)\Big|_0^2$$

$$= 2\pi(8-8/3)$$

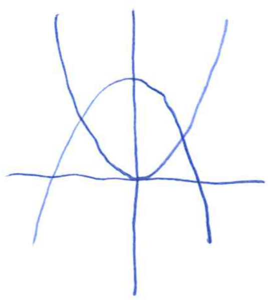
$$= 32\pi/3 \quad \#$$

eg. Let Ω be the solid bdd between $z = 8-x^2-y^2$ and $z = x^2+3y^2$. Find its volume.

The 2 surfaces intersect at

$$8-x^2-y^2 = z = x^2+3y^2, \text{ ie}$$

$$x^2+2y^2 = 4$$



Cross section

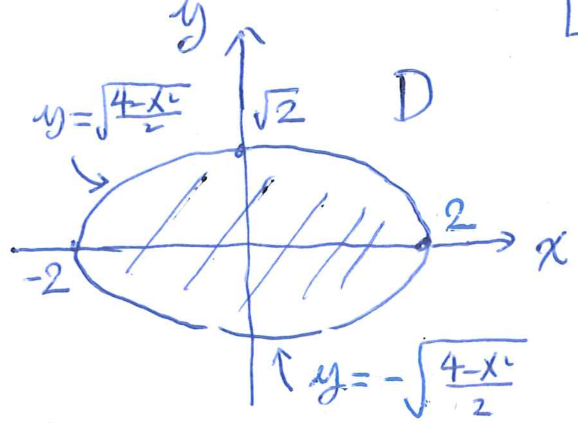
$\therefore\ \Omega:\quad x^2+3y^2 \le z \le 8-x^2-y^2$

(x,y) satisfies $x^2+2y^2 \le 4$.

$$\text{vol} = \iiint_\Omega 1\,dV$$

$$= \iint_D \int_{x^2+3y^2}^{8-x^2-y^2} 1\,dz\,dA(x,y)$$

$$= \iint_D (8-2x^2-4y^2)\,dA(x,y)$$

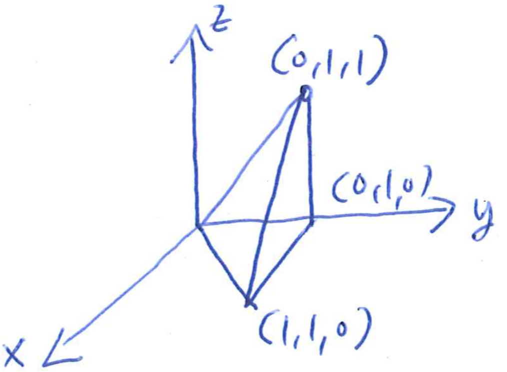


$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx$$

$\vdots$

$= 8\pi\sqrt{2}$ (see Text for details)

e.g. Let T be the tetrahedron with vertices at $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$. Express $\iiint_T f\,dV$ in the order $dz\,dx\,dy$ and then in $dy\,dz\,dx$.



The face passing $(0,0,0)$, $(0,1,1)$, $(1,1,0)$ lies on the plane:

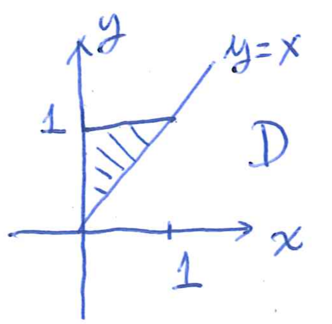
$x-y+z=0$. Use

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i}+\hat{j}-\hat{k}.$$

The other three faces: $z=0$ (bottom), $y=1$ (front), $x=0$ (back).

T can be expressed as

$$0 \le z \le -x+y$$
$$(x,y)\in D$$

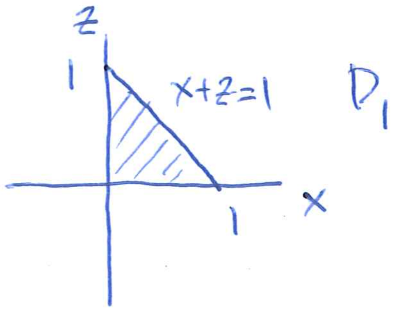


$$\therefore \iiint_T f\,dV = \iint_D \int_0^{-x+y} f(x,y,z)\,dz\,dA(x,y)$$

$$= \int_0^1\int_0^y\int_0^{-x+y} f(x,y,z)\,dz\,dx\,dy.$$

As graphs over xz-plane, T can be described as

$$x+z \le y \le 1$$
$$(x,y) \in D_1$$



$$\therefore \iiint_T f\,dV = \iint_{D_1} \int_{x+z}^{1} f(x,y,z)\,dy\,dA(x,z)$$
$$= \int_0^1 \int_0^{1-x} \int_{x+z}^{1} f(x,y,z)\,dy\,dz\,dx \quad \#$$

(It is $\int_0^1 \int_z^1 \int_0^{y-z} f(x,y,z)\,dx\,dy\,dz$ in the order $dx\,dy\,dz$ — Exercise)